Solution for HW9

1-12-2016

§6.6] 1) For $|z|<1$, we have

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n$$

$$\Rightarrow \frac{d}{dz}\left(\frac{1}{1-z}\right)=\frac{d}{dz}\sum_{n=0}^{\infty} z^n$$

$$\frac{1}{(1-z)^2}=\sum_{n=1}^{\infty} n z^{n-1}$$

$$=\sum_{n=0}^{\infty}(n+1)z^n$$

$$\Rightarrow \frac{d}{dz}\frac{1}{(1-z)^2}=\frac{d}{dz}\sum_{n=0}^{\infty}(n+1)z^n$$

$$\frac{2}{(1-z)^3}=\sum_{n=0}^{\infty}(n+1)(n)z^{n-1}$$

$$=\sum_{n=1}^{\infty}(n+1)(n)z^{n-1}$$

$$=\sum_{n=0}^{\infty}(n+2)(n+1)z^n.$$

2) Since $1<|z-1|<+\infty$, we have $\frac{1}{|z-1|}<1$.

So $$\frac{1}{\left(1-\frac{1}{1-z}\right)^2}=\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{1-z}\right)^n$$

$$\frac{(1-z)^2}{z^2}=\sum_{n=0}^{\infty}\frac{(n+1)}{(1-z)^n}$$

$$\frac{1}{z^2}=\sum_{n=0}^{\infty}\frac{(n+1)}{(1-z)^{n+2}}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(z-1)^{n+2}}$$

$$=\sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{(z-1)^n}$$

5) At $z_0 = \frac{\pi}{2}$,

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right)$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \left(z-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$$

For $z \neq \frac{\pi}{2}$, we have

$$f(z) = \frac{\cos z}{\left(z+\frac{\pi}{2}\right)\left(z-\frac{\pi}{2}\right)}$$

$$= \frac{-1}{\left(z+\frac{\pi}{2}\right)} \sum_{n=0}^{\infty} \frac{(-1)^n \left(z-\frac{\pi}{2}\right)^{2n}}{(2n+1)!}$$

$$= \frac{-1}{\left(z+\frac{\pi}{2}\right)} \left(1 - \frac{\left(z-\frac{\pi}{2}\right)^2}{3!} + \frac{\left(z-\frac{\pi}{2}\right)^4}{5!} - \cdots\right)$$

Note that R.H.S. is also well-defined at $z=\frac{\pi}{2}$ with value $\frac{-1}{\pi}$.
Hence f has a Talyor series expansion at $z=\frac{\pi}{2}$ and f is analytic there.

Similarly, at $z_0 = -\frac{\pi}{2}$,

$$\cos z = \sin\left(z+\frac{\pi}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(z+\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$$

For $z \neq -\frac{\pi}{2}$, we have

$$f(z) = \frac{\cos z}{\left(z-\frac{\pi}{2}\right)\left(z+\frac{\pi}{2}\right)}$$

$$= \frac{1}{z-\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(z+\frac{\pi}{2}\right)^{2n}}{(2n+1)!}$$

Note that R.H.S. is well-defined at $z=-\frac{\pi}{2}$ with value $\frac{-1}{\pi}$.
Hence f has a Talyor series expansion at $z=\frac{\pi}{2}$ and f is analytic there.

6) For $|w-1|<1$, we have

$$\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n$$

So for $|z-1|<1$,

$$\int_1^z \frac{1}{w}\,dw = \int_1^z \sum_{n=0}^{\infty} (-1)^n (w-1)^n \,dw$$

$$\log z - \text{Log}\, 1 = \sum_{n=0}^{\infty} (-1)^n \frac{(w-1)^{n+1}}{n+1}\Bigg|_1^z$$

$$\text{Log}\, z = \sum_{n=0}^{\infty} \frac{(-1)^n (z-1)^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-1)^n}{n}$$

§67) 1) $$\frac{e^z}{z(z^2+1)} = \frac{1}{z}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(1-z^2+z^4-z^6+\cdots\right)$$

$$= \frac{1}{z}(1)(1) + \frac{1}{z}(z)(1) + \frac{1}{z}\left[(1)(-z^2)+\left(\frac{z^2}{2!}\right)(1)\right]$$

$$+ \frac{1}{z}\left[\left(\frac{z^3}{3!}\right)(1)+(z)(-z^2)\right] + \cdots$$

$$= \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots$$

3) $$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$\frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^4 + \cdots$$

$$z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots \;\Big)\; 1$$

$$1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \frac{z^4}{5!} + \cdots$$

$$-\frac{z}{2!} - \frac{z^2}{3!} - \frac{z^3}{4!} - \frac{z^4}{5!} - \cdots$$

$$-\frac{z}{2} - \frac{z^2}{4} - \frac{z^3}{12} - \frac{z^4}{48} - \cdots$$

$$\frac{z^2}{12} + \frac{z^3}{24} + \frac{z^4}{80}$$

$$\frac{z^2}{12} + \frac{z^3}{24} + \frac{z^4}{72} + \cdots$$

$$-\frac{z^4}{720} + \cdots$$

$$-\frac{z^4}{720} + \cdots$$

§71) 1) a) $\frac{1}{z+z^2} = \frac{1}{z}\left(\frac{1}{1+z}\right) = \frac{1}{z}(1-z+z^2-z^3+\cdots) = \frac{1}{z}-1+z+\cdots$

for $|z|<1$.

Hence $\underset{z=0}{\mathrm{Res}}\left(\frac{1}{z+z^2}\right) = 1$

b) For $0<|z|<+\infty$,

$z\cos\frac{1}{z} = z\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{z}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{-2n+1} = z-\frac{1}{2z}+\cdots$

Hence $\underset{z=0}{\mathrm{Res}}\left(z\cos\frac{1}{z}\right) = \frac{-1}{2}$.

c) For $0<|z|<+\infty$,

$$\frac{z-\sin z}{z} = \frac{1}{z}\left(z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\right)$$

$$= \frac{1}{z}\left(\frac{z^3}{3!}-\frac{z^5}{5!}+\cdots\right)$$

$$= \frac{z^2}{3!}-\frac{z^4}{5!}+\cdots$$

Hence $\underset{z=0}{\mathrm{Res}}\left(\frac{z-\sin z}{z}\right) = 0$

2) b) By Cauchy's Integral formula,

$$\int_{|z|=3}\frac{e^{-z}}{(z-1)^2}dz = 2\pi i\left.\frac{d}{dz}e^{-z}\right|_{z=1} = \frac{-2\pi i}{e}.$$

d) $$\int_{|z|=3}\frac{z+1}{z(z-2)}dz = \int_{|z|=3}\frac{-1}{2z}dz + \int_{|z|=3}\frac{3}{2(z-2)}dz$$

$$= 2\pi i\left(\frac{-1}{2}+\frac{3}{2}\right)$$

$$= 2\pi i.$$

3) a) Let $f(z) = \frac{z^5}{1-z^3}$.

Note that the singularities of $f(z)$ are 1, $e^{\frac{2\pi i}{3}}$ and $e^{\frac{4\pi i}{3}}$, which lie inside the circle $|z|=2$.

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{z^{-5}}{1-z^{-3}} = \frac{-1}{z^4} \cdot \frac{1}{1-z^3}$$

$$\Rightarrow \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{-1}{z^4}(1+z^3+z^6+\dots) = \frac{-1}{z^4} - \frac{1}{z} - z^2 - \dots$$

Hence $\int_{|z|=2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -2\pi i$ //

b) Let $g(z) = \frac{1}{1+z^2}$.

Note that the singularities of $f(z)$ are $\pm i$, which lie inside the circle $|z|=2$.

$$\frac{1}{z^2} g\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{1}{1+z^{-2}} = \frac{1}{1+z^2} = 1 - z^2 + z^4 - \dots$$

Hence $\int_{|z|=2} g(z)\,dz = 2\pi i \operatorname{Res}\left(\frac{1}{z^2} g\left(\frac{1}{z}\right)\right) = 0$.